PERGAMON

Available online at www.sciencedirect.com

SCIENCE DIRECT®

Computers and Structures 81 (2003) 331–338

Computers & Structures

www.elsevier.com/locate/compstruc

Empirical modelling of shear strength of RC deep beams by genetic programming

A.F. Ashour *, L.F. Alvarez, V.V. Toropov

School of Engineering, Design and Technology, University of Bradford, West Yorkshire BD7 1DP, UK

Received 22 June 2001; accepted 1 November 2002

Abstract

This paper investigates the feasibility of using genetic programming (GP) to create an empirical model for the complicated non-linear relationship between various input parameters associated with reinforced concrete (RC) deep beams and their ultimate shear strength. GP is a relatively new form of artificial intelligence, and is based on the ideas of Darwinian theory of evolution and genetics. The size and structural complexity of the empirical model are not specified in advance, but these characteristics evolve as part of the prediction. The engineering knowledge on RC deep beams is also included in the search process through the use of appropriate mathematical functions.

The model produced by GP is constructed directly from a set of experimental results available in the literature. The validity of the obtained model is examined by comparing its response with the shear strength of the training and other additional datasets. The developed model is then used to study the relationships between the shear strength and different influencing parameters. The predictions obtained from GP agree well with experimental observations.

Keywords: Reinforced concrete deep beams; Genetic programming; Empirical model building

1. Introduction

Reinforced concrete (RC) deep beams are characterised as being relatively short and deep, having a thickness that is small relative to their span or depth, and being primarily loaded in their own plane. They are sometimes used for load distribution, for example as transfer girders, pile caps, folded plates and foundation walls. The transition from RC shallow beam behaviour to that of deep beams is imprecise. For example, while the ACI code [1], CEB-FIP model code [5] and CIRIA Guide 2 [6] use the span/depth ratio limit to define RC deep beams, the Canadian code [7] employs the concept of shear span/depth ratio. ACI defines beams with clear span to effective depth ratios less than 5 as deep beams, whereas CEB-FIP model code treats simply supported and continuous beams of span/depth ratios less than 2 and 2.5, respectively, as deep beams.

Several possible modes of failure of deep beams have been identified from physical tests but due to their geometrical dimensions shear strength appears to control their design. Despite of the large amount of research carried out over the last century, there is no agreed rational procedure to predict the shear strength of RC deep beams [12,22]. This is mainly because of the very complex mechanism associated with the shear failure of RC beams.

The design of RC deep beams has not yet been covered by BS8110 [4] that explicitly states, “for the design of deep beams, reference should be made to specialist literature”. Comparisons between test results and predictions from other codes, such as ACI and CIRIA Guide 2, show poor agreement [28,30].

In this paper, the genetic programming (GP) method is used to build an empirical model to estimate the shear

* Corresponding author. Tel.: +44-1274-233870; fax: +44-1274-233888.

E-mail address: a.f.ashour@bradford.ac.uk (A.F. Ashour).

0045-7949/03/$ - see front matter © 2003 Elsevier Science Ltd. All rights reserved.
doi:10.1016/S0045-7949(02)00437-6

strength of RC deep beams subjected to two point loads. The GP model will directly evolve from a set of experimental results available in the literature. A parametric study is conducted to examine the validity of the GP model predictions.

2. Overview of the genetic programming methodology

GP [13] is a branch of genetic algorithms (GAs). Its basis is the same Darwinian concept of survival of the fittest. While a GA uses a string of numbers to represent the solution, the GP creates a population of computer programs with a tree structure. In this application, a program represents an empirical model to be used for approximation of the shear strength of RC deep beams. A typical program, representing the expression $(x_1/x_2 + x_3)^2$, is shown in Fig. 1.

These randomly generated programs are general and hierarchical, varying in size and structure. GP's main goal is to solve a problem by searching highly fit computer programs in the space of all possible programs. This aspect is the key to finding near global optimum solutions by keeping many solutions that may potentially be close to minima (local or global). The creation of the initial population is a blind random search of the space defined by the problem. Unlike a GA's output that is a quantity, the output of the GP is a program.

The programs are composed of elements from terminal and functional sets, called nodes (see Fig. 1). The terminal set consists of N variables $x_1, x_2, \ldots, x_N$. The functional set contains the mathematical operators that will be used to generate the regression model, e.g. {+, ∗, /, power, square, square root, negation,...}. The functional set can be further subdivided into binary nodes, which take any two arguments (like addition), and unary nodes, which take one argument, e.g. a square root. All the functions and terminals must be compatible in order

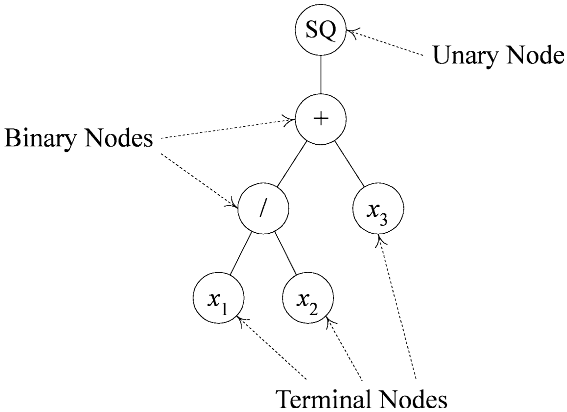

Fig. 1. Typical tree structure for $((x_1/x_2) + x_3)^2$.

to faultlessly pass information between each other (closure property).

3. Genetic operators

Model structures evolve through the action of three basic genetic operators: reproduction, crossover and mutation. In the reproduction stage, a strategy must be adopted as to which programs should die. In this implementation, a small percentage of the trees with the worst fitness are killed. The population is then filled with the surviving trees according to accepted selection mechanisms, as explained below. Crossover swaps randomly selected parts of two trees to combine good information from the parents and to improve the fitness of the next generation (see Fig. 2). Mutation, as shown in Fig. 3, protects the model against premature convergence and improves the non-local properties of the search. Occasionally, one randomly selected node is replaced by another one from the same set, except itself.

An additional operator, elite transfer, is used to allow a relatively small number of the fittest programs, called the elite, to be transferred unchanged to a next generation, in order to keep the best solutions found so far. As a result, a new population containing the same number of trees as the original one is created but it has a higher average fitness value. Details of the particular implementation of GP used in this study are described below.

3.1. Fitness function

When selecting randomly a tree to perform any genetic operation, the tournament selection is used. This method specifies the probability of selection on the basis of the fitness of the solution. The fitness of a solution reflects the quality of approximation of experimental data by a current expression represented by a tree. Another factor to be considered in the definition of the fitness is the length of the tree (compact expressions are desirable) which, in the current implementation, is limited by the maximum allowed number of tuning parameters (function coefficients) allocated to the tree. In problems of empirical model building, an obvious choice for the estimation of the quality of the model is the sum of squares of the difference between the empirical model output and some selected experimental data.

Generally, there can be two sources of error on the empirical model: incorrect structure and inaccurate tuning parameters. In order to separate these errors, the measure $Q(S_i)$ of the quality for a given empirical model S_i is only calculated for the tuned expression. In a dimensionless form this measure of quality of the solution can be presented as follows:

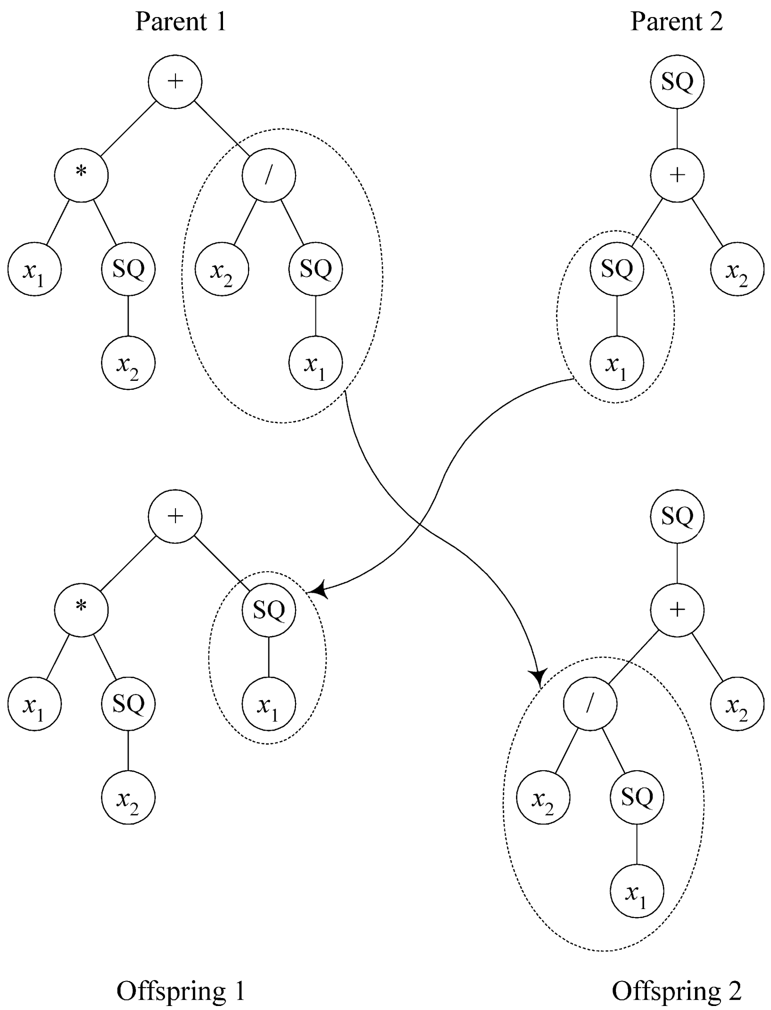

Fig. 2. Crossover.

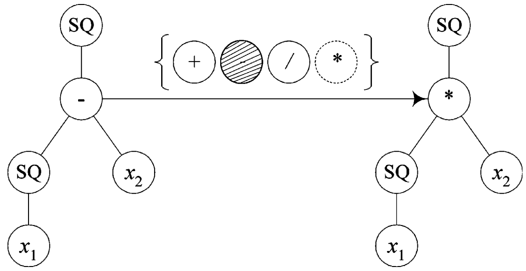

Fig. 3. Mutation.

$$Q(S_i) = \frac{\sum_{p=1}^{P}(F_\mathrm{p} - \widetilde{F}_\mathrm{p})^2}{\sum_{p=1}^{P} F_\mathrm{p}^2} \tag{1}$$

where, for a given tree, $\widetilde{F}_\mathrm{p} = \widetilde{F}(x_\mathrm{p})$ is the predicted value of the shear strength by the empirical model after tuning (see next section) at the pth point of the experimental data, F_p is the experimental value at the same point and P is the total number of experiments. The fitness function $Q(S_i)$ is to be minimized by the evolutionary process.

4. Model tuning

The empirical model is characterised not only by its structure (to be found by the GP) but also by a set of tuning parameters $\boldsymbol{a}$ to be found by the model tuning, i.e. the least square fitting of the model into the set of experimental values:

$$\sum_{p=1}^{P}(F_\mathrm{p} - \widetilde{F}_\mathrm{p}(\boldsymbol{a}))^2 \rightarrow \min \tag{2}$$

To solve this optimization problem, a combination of a GA to find an initial guess followed by a gradient-based optimization method [15] is used.

5. Parameters affecting shear strength of deep beams

Fig. 4 shows the geometrical dimensions and reinforcement of a typical RC deep beam tested under two point loads. The main parameters influencing the shear strength of RC deep beams are the concrete compressive strength, main longitudinal top and bottom steel reinforcement, horizontal and vertical web steel reinforcement, beam width and depth, shear span and beam span [3,10,24]. Those parameters can be expressed in normalised form as follows:

- Normalised shear strength $\lambda = P/bhf'_c$, where $P =$ shear failure load, $b =$ beam width, $h =$ overall beam depth, $f'_c =$ concrete compressive strength.
- Shear span to depth ratio $x_1 = a/h$.
- Beam span to depth ratio $x_2 = L/h$.
- Smeared vertical web reinforcement ratio $x_3 = A_{sv}f_{yv}/bs_v f'_c$, where $A_{sv} =$ area of vertical web reinforcement, $s_v =$ horizontal spacing of vertical web reinforcement, $f_{yv} =$ yield stress of vertical web reinforcement.
- Smeared horizontal web reinforcement ratio $x_4 = A_{sh}f_{yh}/bs_h f'_c$, where $A_{sh} =$ area of horizontal web reinforcement, $s_h =$ vertical spacing of horizontal web reinforcement, $f_{yh} =$ yield stress of horizontal web reinforcement.
- Main longitudinal bottom reinforcement ratio $x_5 = A_{sb}f_{yb}/bhf'_c$, where $A_{sb} =$ area of main longitudinal bottom reinforcement, $f_{yb} =$ yield stress of main longitudinal bottom reinforcement.

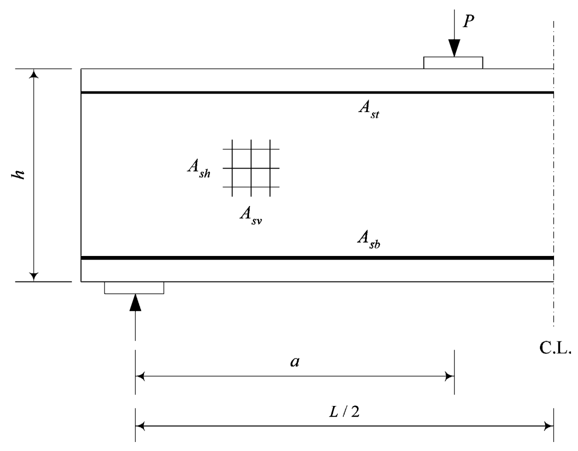

Fig. 4. Geometrical dimensions of a RC deep beam.

- Main longitudinal top reinforcement ratio $x_6 = A_{st}f_{yt}/bhf'_c$, where $A_{st} =$ area of main longitudinal top reinforcement, $f_{yt} =$ yield stress of main longitudinal top reinforcement.

Following normal practice established in the majority of papers published on this topic [18,31], the transformation of the physical variables into dimensionless parameters allowed the reduction of the initial set of 16 variables to only six dimensionless variables. A dimensionless format is typically used in the codes of practice and can be easily understood by design engineers. In addition, the dimensionless transformation of the initial physical variables has not been done arbitrarily but follows design expertise of structural engineers, i.e. $x_1 = a/h$ could have been defined as $x_1 = a/L$ but this would not make sense to a design engineer; x_3 and x_4 define the smeared intensity of vertical and horizontal web reinforcement. In applications where the number of variables is small, the response function produced by GP could be directly related to the physical variables, as suggested by Keijzer and Babovic [8,9] who developed a dimensionally aware GP.

The shear span to depth ratio x_1 is one of the main parameters influencing shear behaviour [3,17,20,24]. A marked increase in the shear strength occurs in RC beams with reducing the shear span to depth ratio. The type of web reinforcement affects the shear strength of RC deep beams [10,23]. Most codes of practice provide formulae to calculate the shear strength in which the contribution of the horizontal web reinforcement is higher than that of the vertical web reinforcement. Leonhardt and Walther [14] suggested that the shear strength of deep beams cannot be improved by the addition of web reinforcement. However, Kong et al. [10] suggested that improvement is possible to a limited extend. Rogowsky et al. [23] concluded that horizontal web reinforcement had no effect on the shear strength while the vertical web reinforcement had a significant influence. As explained above, there is strong disagreement on the influence of web reinforcement on the shear strength of deep beams, in particular the relative effectiveness of vertical and horizontal reinforcement. Although most test results of RC deep beams suggest that the span to depth ratio x_2 has very little influence on shear strength [18,26,31], most codes of practice use this parameter to define deep beams. In the current research, the span to depth ratio will be represented in the GP model as one variable and its effect on the shear strength will be studied. Using the current technique, it will be possible to study the effect of all parameters on the ultimate shear strength of deep beams using all test results available in the literature at the same time; this may eliminate the inconsistency and conflicting conclusions drawn by different researches.

6. Empirical model obtained by GP

There is a large number of test results of RC deep beams referred to in the literature. Test results of 141 deep beams reported in [10,11,17,20,21,23–25,27,29] are used to create the GP response. The training dataset covers a wide range of each parameter as given in Table 1. All selected beams were tested under two point loads; other load arrangements have been excluded.

The mathematical operators addition, multiplication, division, square and negation and a population size of 500 individuals were selected in the initial runs. For simplicity of the GP evolved expression, the power of variables was restricted to positive integer values.

From the beginning, it was observed that the variable x_2 (beam span to depth ratio) had small influence on the shear strength λ and, on one occasion, GP did not include this variable in the evolved expression.

To confirm this observation, several GP runs were undertaken with the fitness function given in Eq. (1) replaced by the statistical concept of correlation [19] as defined in Eq. (3) below:

$$Q(S_i) = \left| \frac{\sum_{p=1}^{P}(\widetilde{F}_p - \overline{\overline{F}})(F_p - \overline{F})}{\sqrt{\sum_{p=1}^{P}(\widetilde{F}_p - \overline{\overline{F}})^2}\sqrt{\sum_{p=1}^{P}(F_p - \overline{F})^2}} \right|, \quad 0 \leqslant Q(S_i) \leqslant 1 \tag{3}$$

where, for a given tree, $\overline{\overline{F}}$ is the mean of GP predicted function values over the P points in the experimental data, and, similarly, $\overline{F}$ is the mean of the experimental shear strength values over all experimental data. The fitness function given in (3) determines and quantifies the correlation between the independent variables (x_1, x_2, x_3, x_4, x_5, x_6) and the dependant variable λ. The closer the fitness value to 1, the stronger the correlation. In all GP runs, the fitness value $Q(S_i)$ in (3) was close to 1 when variable x_2 was not included in the final GP expression. The small relevance of x_2 on the shear strength has also been experimentally observed by other researchers [3,11,26,28,31].

Table 1
Range of normalised function and parameters of the training dataset

	Minimum	Maximum
x_1	0.28	2.0
x_2	0.9	4.14
x_3	0.0	0.32
x_4	0.0	0.21
x_5	0.023	0.445
x_6	0.0	0.128
$\lambda(x)$	0.029	0.308

In the next stage, only variables x_1, x_3, x_4, x_5 and x_6 were used. Multiple runs were performed and the solutions analysed on the basis of the simplest generated model that conformed as closely as possible to the engineering understanding of the failure mechanism. When the population size was increased to 1000 individuals and the mutation rate set to 0.001, the following model emerged:

$$\lambda = x_5 * (4.31 + 0.15 * x_1^2 + 12.11 * x_1 * x_5 + 3.34 * x_1 * x_6 + 0.66 * x_3 + 0.47 * x_4 + 23.27 * x_5^2 - 16.97 * x_1 * x_5^2 - 18.22 * x_5 - 2.70 * x_1) \tag{4}$$

Solutions with better fitness than (4) were produced, but they were rejected because of their excessive length. Simplicity is a requirement and, as the complexity of the model increases, its ability to generalise can be affected by the risk of overfitting the data.

The structure of expression (4) was found acceptable, but the coefficients needed to be adjusted in order to satisfy some constraints derived from the engineering knowledge of the problem, such as that the shear strength should be positive for the range of shear span to depth ratio studied. A sequential quadratic programming (SQP) algorithm [16] was applied to improve the coefficients of Eq. (4), as follows:

$$\lambda = x_5 * (3.50 + 0.20 * x_1^2 + 3.3 * x_1 * x_5 + 3.37 * x_1 * x_6 + 0.63 * x_3 + 0.71 * x_4 + 9.8 * x_5^2 - 1.66 * x_1 * x_5^2 - 10.67 * x_5 - 1.76 * x_1) \tag{5}$$

Further studies with GP and manual postprocessing to adjust the coefficients produced by the SQP algorithm have suggested a simplified final expression as follows:

$$\lambda = A * x_5^2 + B * x_5 + C \tag{6}$$

where $A = -4.56 + 1.68 * x_1$, $B = 2.45 + 0.1 * x_1^2 - 1.16 * x_1 + 3.12 * x_6$, $C = 0.3 * x_3 + 0.3 * x_4$.

It appeared that the variables x_1 (shear span to depth ratio) and x_5 (main longitudinal bottom reinforcement ratio) were the most significant parameters. Alternative expressions with an additional term $x_1 * x_6 * x_5$ were found, but no relationship between these variables is available as a criterion for the choice between different acceptable expressions. In the literature there is no consensus about the effect of the main longitudinal top reinforcement (represented by x_6 in the above expression) on the shear strength; this requires further investigation and, following that, better understanding of its effect can be reflected in the GP prediction. The web reinforcement contribution (represented by x_3 and x_4) as given by expression (6) is very small.

Expression (6) gives a root mean square (RMS) error over the training data of 0.033. The average ratio between the predicted and experimental shear strength is 1.008, and the standard deviation is 0.23. Fig. 5 shows a comparison between the experimental and predicted shear strengths for the training data. To validate the model, 15 additional experimental results, which were not exposed to GP in the training process, were used and detailed in Table 2. The average and standard deviation of the ratio between the predicted and experimental shear strengths are 1.11 and 0.21, respectively. The RMS error over the validation data is 0.035.

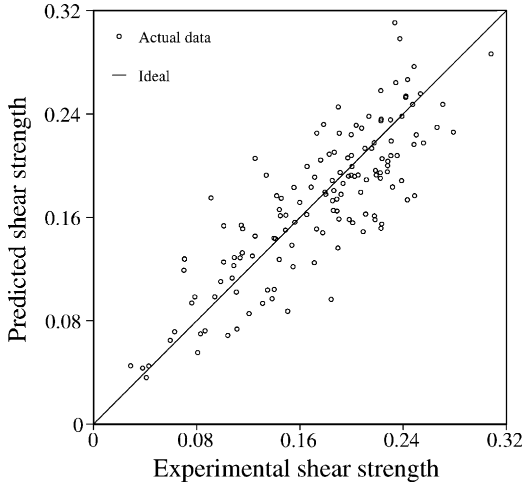

Fig. 5. Comparison of experimental and predicted shear strengths.

7. Parametric study

The influence of different parameters on the shear strength λ is studied next. In this study, expression (6) has been used to predict the shear strength outside the range where the model was constructed. Parameter x_1 is predicted up to the value 2.5 and parameter x_5 starts from the value 0 (see Table 1 for the initial range of the parameters of the training dataset).

Fig. 6 gives the relationship between the shear span to depth ratio x_1 against the shear strength λ for different values of the normalised main longitudinal bottom reinforcement ($x_5 = 0.025,\ 0.05,\ 0.075,\ 0.1,\ 0.2$) and no

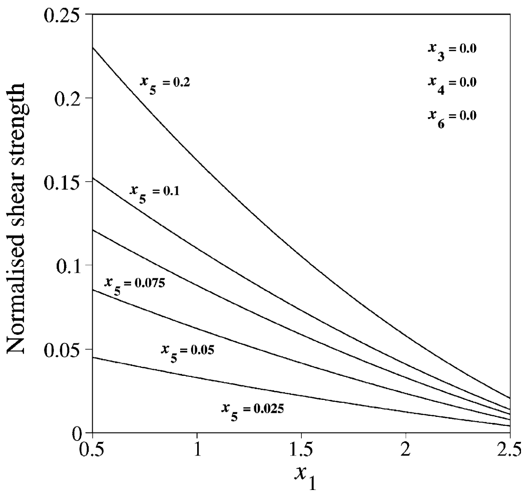

Fig. 6. Effect of x_1 on the shear strength.

Table 2
Validation dataset

Ref.	x_1	x_3	x_4	x_5	x_6	λ_e	λ	λ/λ_e
[20]	0.8889	0.0843	0.0000	0.1413	0.0834	0.2030	0.2125	1.0467
[10]	0.5000	0.0855	0.0855	0.1049	0.0256	0.2214	0.2175	0.9825
[17]	0.5793	0.0000	0.0000	0.0904	0.0000	0.1542	0.1345	0.8723
[21]	0.3776	0.0334	0.0334	0.0743	0.0000	0.1390	0.1489	1.0713
[23]	1.6667	0.0000	0.0239	0.1071	0.0064	0.0684	0.0742	1.0849
[24]	0.8571	0.1342	0.1450	0.3519	0.0192	0.2339	0.2566	1.0967
[25]	0.9389	0.0626	0.1033	0.1222	0.0475	0.1442	0.2004	1.3903
[29]	0.5029	0.0141	0.0141	0.2824	0.0028	0.1568	0.2489	1.5881
[27]	1.2324	0.0000	0.0000	0.2899	0.0000	0.1190	0.1306	1.0977
[27]	1.2324	0.0000	0.0000	0.2899	0.0000	0.1590	0.1306	0.8215
[27]	1.2324	0.0000	0.0000	0.2899	0.0000	0.1640	0.1306	0.7965
[2]	1.0000	0.0417	0.0833	0.1395	0.0065	0.1473	0.1781	1.2093
[2]	1.0000	0.0417	0.0208	0.1395	0.0065	0.1232	0.1594	1.2943
[2]	1.0000	0.0491	0.0245	0.1644	0.0548	0.1645	0.2009	1.2210
[2]	1.0000	0.0000	0.0000	0.1644	0.0548	0.1634	0.1788	1.0943
Average								1.11
Standard deviation								0.21
RMS								0.035

web reinforcement. A significant reduction on the dimensionless shear strength λ is observed with the increase of the shear span to depth ratio x_1; the rate of decrease is decreased with the increase of shear span to depth ratio. This behaviour was experimentally and computationally observed by other researchers [3,10, 18,24].

The influence of the main longitudinal bottom reinforcement ratio x_5 on the dimensionless shear strength λ is presented in Fig. 7 for different shear span to depth ratios ($x_1 = 0.5$, 1.0, 1.5, 2.0, 2.5) and no web reinforcement. For all cases, the dimensionless shear strength is increasing with the increase of main longitudinal bottom reinforcement ratio. The rate of increase of the shear strength is reduced with the increase in the main longitudinal reinforcement ratio. These observations agree well with conclusions obtained elsewhere [3,10,18,24].

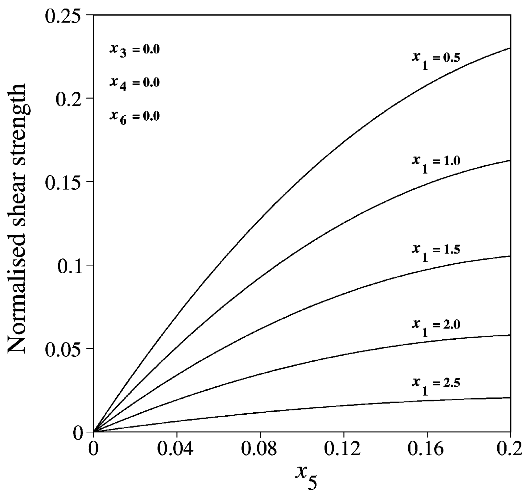

Fig. 7. Effect of x_5 on the shear strength.

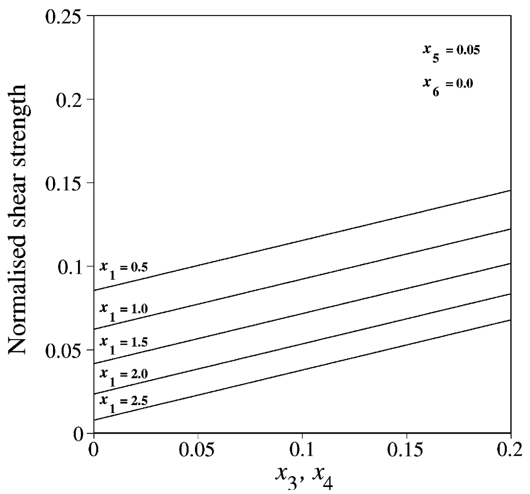

Fig. 8. Effect of x_3 and x_4 on the shear strength.

Fig. 8 presents the influence of the horizontal and vertical web reinforcement ratios on the shear strength. Based on the current model (Eq. (6)), the contribution of the web reinforcement to the shear strength is small and more pronounced for smaller shear span to depth ratio. Both smeared vertical and horizontal web reinforcement ratios have the same contribution to the shear strength, as indicated by Eq. (6).

8. Conclusions

An empirical model to predict the shear strength of RC deep beams has been obtained by GP. Experimental results are used to build and validate the model. Good agreement between the model predictions and experiments has been achieved. As more experimental results and knowledge of the shear behaviour of deep beams become available, the GP prediction could be improved.

The GP model predicts the following behaviour between the shear strength and the influencing parameters:

- The shear span to depth and main longitudinal bottom reinforcement ratios have the most significant effect on the shear strength of RC deep beams.
- The shear strength is inversely proportional to the shear span to depth ratio; the higher the shear span to depth ratio, the less the shear strength.
- The shear strength increases with the increase of the main longitudinal bottom reinforcement ratio up to a certain limit beyond which no improvement can be achieved.
- The effect of the beam span to depth ratio and web reinforcement on the shear strength is very small.

References

[1] ACI Committee 318. Building code requirements for structural concrete (ACI 318-99) and commentary (ACI 318R-99). Detroit: American Concrete Institute; 1999.

[2] Aguilar G, Matamoros AB, Parra-Montesinos GJ, Ramírez JA, Wight JK. Experimental evaluation of design procedures for shear strength of deep reinforced concrete beams. ACI Struct J 2002;99:539–48.

[3] Ashour AF. Shear capacity of reinforced concrete deep beams. Struct Eng J ASCE 2000;126:1045–52.

[4] British Standards Institution (BSI). Structural use of concrete. Milton Keynes: BSI; BS8110: Part 1, 1997.

[5] CEB-FIP. Model code 1990 for concrete structures. Thomas Telford Services, Ltd, London, Comité Euro-International du Béton, Laussane, 1993. p. 437.

[6] CIRIA Guide 2. The design of deep beams in reinforced concrete. London: Over Arup and Partners, and

Construction Industry Research and Information Association; 1977. p. 131. Reprinted 1984.
[7] CSA Technical Committee on Reinforced Concrete Design, A23.3-94. Design of concrete structures. Ontario, Canada: Canadian Standards Association; 1994. p. 199.
[8] Keijzer M, Babovic V. Dimensionally aware genetic programming. In: Banzhaf W, Daida J, Eiben AE, Garzon MH, Honavar V, Jakiela M, editors. GECCO-99: Proceedings of the Genetic and Evolutionary Computation Conference, Orlando, Florida, USA. Morgan Kauffman Publishers; 1999.
[9] Keijzer M, Babovic V. Genetic programming within a framework of computer-aided discovery of scientific knowledge. In: Whitley D, Goldberg D, Cantu-Paz E, Spector L, Parmee I, Beyer H, editors. GECCO-2000: Proceedings of the Genetic and Evolutionary Computation Conference, Las Vegas, USA. Morgan Kauffman Publishers; 2000.
[10] Kong FK, Robins PJ, Cole DF. Web reinforcement effects on deep beams. ACI J 1970;67:1010–7.
[11] Kong FK, Robins PJ, Kirby DP, Short DR. Deep beams with inclined web reinforcement. ACI J 1972;69:172–6.
[12] Kong FK. In: Kong FK, editor. Reinforced concrete deep beams. New York: Blackie; 1990.
[13] Koza JR. Genetic programming: on the programming of computers by means of natural selection. London: MIT Press; 1992.
[14] Leonhardt F, Walther R. Deep beams. Deutscher Ausschuss Für Stahlbeton Bulletin 178, Wilhelm Ernst and Sohn (Berlin), CIRIA English Translation, 1970.
[15] Madsen K, Hegelund P. Non-gradient subroutines for non-linear optimisation. Institute for Numerical Analysis, Technical University of Denmark, Report NI-91-05, 1991.
[16] Madsen K, Tingleff O. Robust subroutines for non-linear optimisation. Institute for Numerical Analysis, Technical University of Denmark, Report NI-90-06, 1990.
[17] Manuel RF, Slight BW, Suter GT. Deep beam behaviour affected by length and shear span variations. ACI J 1971;68:954–8.
[18] Mau ST, Hsu TTC. Shear strength prediction for deep beams with web reinforcement. ACI Struct J 1987;84:513–23.
[19] McKay B, Willis MJ, Hidden HG, Montague GA, Barton GW. Identification of industrial processes using genetic programming. In: Friswell MI, Mottershead JE, editors. Proceedings of International Conference on Identification in Engineering Systems, Swansea. The Cromwell Press Ltd; 1996. p. 328–37.
[20] De Paiva HA, Siess CP. Strength and behaviour of deep beams in shear. J Struct Div, ASCE 1965;ST5:19–41.
[21] Ramakrishnan V, Ananthanarayana Y. Ultimate strength of deep beams in shear. ACI J 1968;65:87–98.
[22] Regan PE. Research on shear: a benefit to humanity or a waste of time?. Struct Eng 1993;71(19):337–46.
[23] Rogowsky DM, MacGregor JG, Ong SY. Test of reinforced concrete deep beams. ACI Struct J 1986;83:614–23.
[24] Smith KN, Vantsiotis AS. Shear strength of deep beams. ACI Struct J 1982;79:201–13.
[25] Subedi NK, Vardy AE, Kubota N. Reinforced concrete deep beams—some test results. Mag Concr Res 1986; 38(137):206–19.
[26] Subedi NK. Reinforced concrete deep beams: a method of analysis. Proceedings of Institution of Civil Engineering, Part 2 1988;85:1–30.
[27] Suter GT, Manuel RF. Diagonal crack width control in short beam. ACI J 1971;68:451–5.
[28] Tan KH, Weng LW, Teng S. A strut-and-tie model for deep beams subjected to combined top-and-bottom loading. Struct Eng J (Lond) 1997;75(13):215–25.
[29] Tan KH, Lu HY. Shear behaviour of large reinforced concrete deep beams and code comparisons. ACI Struct J 1999;96(5):836–45.
[30] Teng S, Kong FK, Poh SP. Shear strength of reinforced and pre-stressed concrete deep beams. Part I: current design methods and a proposed equation. Proc Inst Civil Eng Struct Buildings 1998;128:112–23.
[31] Wang W, Jiang D, Thomas Hsu C. Shear strength of reinforced concrete deep beams. J Struct Eng 1993; 119(8):2294–312.